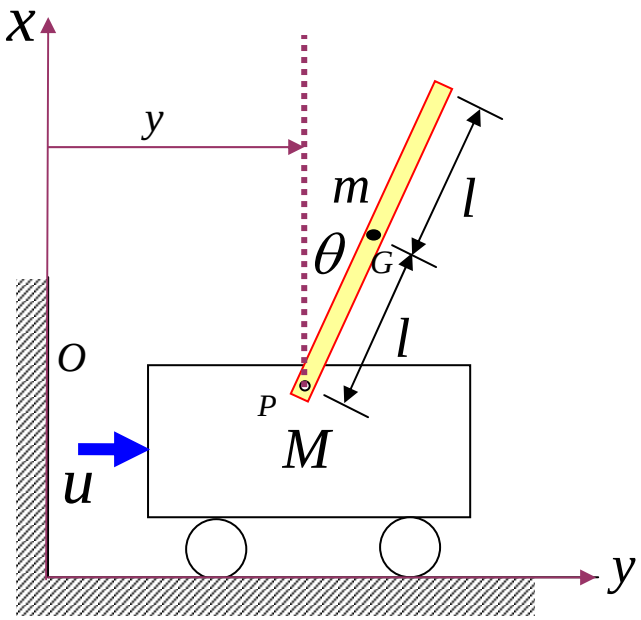
倒立桿之數學模型與 Simulink Model 之建立



$$\vec{a}_G = \vec{a}_P + \vec{\omega}\times\left(\vec{\omega}\times\overrightarrow{PG}\right)+\vec{\alpha}\times\overrightarrow{PG}$$

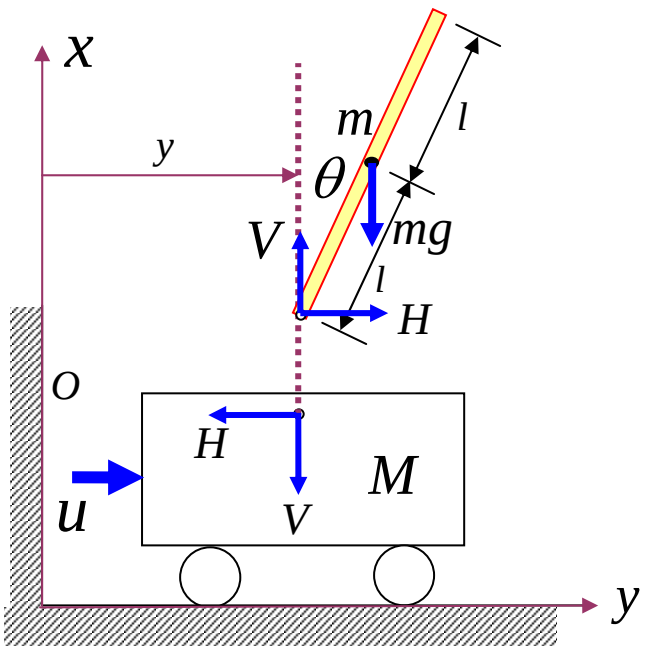
$$\vec{a}_P = \ddot{y}\,\vec{j} \qquad \vec{\omega} = \dot{\theta}\vec{k} \qquad \vec{\alpha} = \ddot{\theta}\vec{k}$$

$$\overrightarrow{PG} = l\cos\theta\,\vec{i} + l\sin\theta\,\vec{j}$$

$$\vec{\omega}\times\left(\vec{\omega}\times\overrightarrow{PG}\right) = -\dot{\theta}^2\left(l\cos\theta\,\vec{i} + l\sin\theta\,\vec{j}\right)$$

$$\vec{\alpha}\times\overrightarrow{PG} = -l\ddot{\theta}\sin\theta\,\vec{i} + l\ddot{\theta}\cos\theta\,\vec{j}$$

$$\vec{a}_G = -l\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)\vec{i} + \left(\ddot{y} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right)\vec{j}$$



The linear motion of center of gravity of pendulum rod is given by

$$\sum \vec{F} = m\vec{a}_G$$

$$(V - mg)\vec{i} + H\,\vec{j} = -ml\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)\vec{i} + m\left(\ddot{y} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right)\vec{j}$$

$$V - mg = -ml\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) \Rightarrow V = mg - ml\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) \quad (1)$$

$$H = m\left(\ddot{y} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right) \quad (2)$$

The rotational motion of center of gravity of pendulum rod can be described by

$$\sum \vec{M}_G = I\vec{\alpha}$$

$$\sum \vec{M}_G = \overrightarrow{GP} \times \left(V\vec{i} + H\,\vec{j}\right), \qquad \overrightarrow{GP} = -l\cos\theta\,\vec{i} - l\sin\theta\,\vec{j}$$

$$\left(Vl\sin\theta - Hl\cos\theta\right)\vec{k} = I\ddot{\theta}\vec{k}$$

$$Vl\sin\theta - Hl\cos\theta = I\ddot{\theta} \quad (3)$$

將(1)、(2)兩式

$$V = mg - ml\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) \quad (1)$$

$$H = m\left(\ddot{y} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right) \quad (2)$$

代入(3)

$$mgl\sin\theta - ml^2\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)\sin\theta - ml\left(\ddot{y} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta\right)\cos\theta = I\ddot{\theta}$$

整理得

$$(ml\cos\theta)\,\ddot{y} + \left(I + ml^2\right)\ddot{\theta} = mgl\sin\theta \quad (4)$$

The horizontal motion of cart is described by

$u - H = M\ddot{y} \Rightarrow M\ddot{y} + H = u$

將上式之 H 以(2)式　$H = m\left(\ddot{y} - l\dot{\theta}^2 \sin\theta + l\ddot{\theta}\cos\theta\right)$ 代入

整理得

$$\left(M + m\right)\ddot{y} + ml\ddot{\theta}\cos\theta = u + ml\dot{\theta}^2 \sin\theta \qquad (5)$$

將(4)、(5)兩式重寫如下

$$(ml\cos\theta)\,\ddot{y} + \left(I + ml^2\right)\ddot{\theta} = mgl\sin\theta \qquad (4)$$

$$\left(M + m\right)\ddot{y} + ml\ddot{\theta}\cos\theta = u + ml\dot{\theta}^2 \sin\theta \qquad (5)$$

聯立解得 $\ddot{y}$ 與 $\ddot{\theta}$ 如下

$$\ddot{y} = \frac{\begin{vmatrix} mgl\sin\theta & I + ml^2 \\ u + ml\dot{\theta}^2 \sin\theta & ml\cos\theta \end{vmatrix}}{\begin{vmatrix} ml\cos\theta & I + ml^2 \\ M + m & ml\cos\theta \end{vmatrix}} = \frac{m^2 l^2 g\sin\theta\cos\theta - ml(I + ml^2)\dot{\theta}^2 \sin\theta - (I + ml^2)u}{m^2 l^2 \cos^2\theta - (M + m)(I + ml^2)}$$

$$\ddot{\theta} = \frac{\begin{vmatrix} ml\cos\theta & mgl\sin\theta \\ M + m & u + ml\dot{\theta}^2 \sin\theta \end{vmatrix}}{\begin{vmatrix} ml\cos\theta & I + ml^2 \\ M + m & ml\cos\theta \end{vmatrix}} = \frac{m^2 l^2 \dot{\theta}^2 \sin\theta\cos\theta + (ml\cos\theta)u - (M + m)mgl\sin\theta}{m^2 l^2 \cos^2\theta - (M + m)(I + ml^2)}$$

其中　$I = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2 \Rightarrow I + ml^2 = \frac{4}{3}ml^2$

為了與 Simulink 中倒立桿之範例比較將 $\ddot{\theta}$ 化成下式

$$\ddot{\theta} = \frac{m^2 l^2 \dot{\theta}^2 \sin\theta\cos\theta + (ml\cos\theta)u - (M+m)mgl\sin\theta}{m^2 l^2 \cos^2\theta - (M+m)(I+ml^2)}$$

$$= \frac{m^2 l^2 \dot{\theta}^2 \sin\theta\cos\theta + (ml\cos\theta)u - (M+m)mgl\sin\theta}{m^2 l^2 \cos^2\theta - (M+m)(\frac{4}{3}ml^2)} \qquad (\because I + ml^2 = \frac{4}{3}ml^2)$$

$$= \frac{ml\dot{\theta}^2 \sin\theta\cos\theta + (\cos\theta)u - (M+m)g\sin\theta}{ml\cos^2\theta - (M+m)(\frac{4}{3}l)} \qquad (\text{分子與分母各除以 } ml)$$

$$= \frac{\frac{(ml\dot{\theta}^2 \sin\theta + u)\cos\theta}{M+m} - g\sin\theta}{l\left(\frac{m}{M+m}\cos^2\theta - \frac{4}{3}\right)} \qquad (\text{分子與分母各除以 } M+m)$$

$$= \frac{g\sin(\theta) + \frac{\cos(\theta)(-u - ml\dot{\theta}^2\sin\theta)}{M+m}}{l\left(\frac{4}{3} - \frac{m}{M+m}\cos^2\theta\right)} \qquad (6)$$

將(5)式 $(M+m)\ddot{y} + ml\ddot{\theta}\cos\theta = u + ml\dot{\theta}^2\sin\theta$
整理得

$$\ddot{y} = \frac{u + ml\dot{\theta}^2\sin\theta - ml\ddot{\theta}\cos\theta}{M+m} \qquad (7)$$

請注意：以上推導 l 為桿長之半

Simulink 之範例即以(6)、(7)二式建立 Model

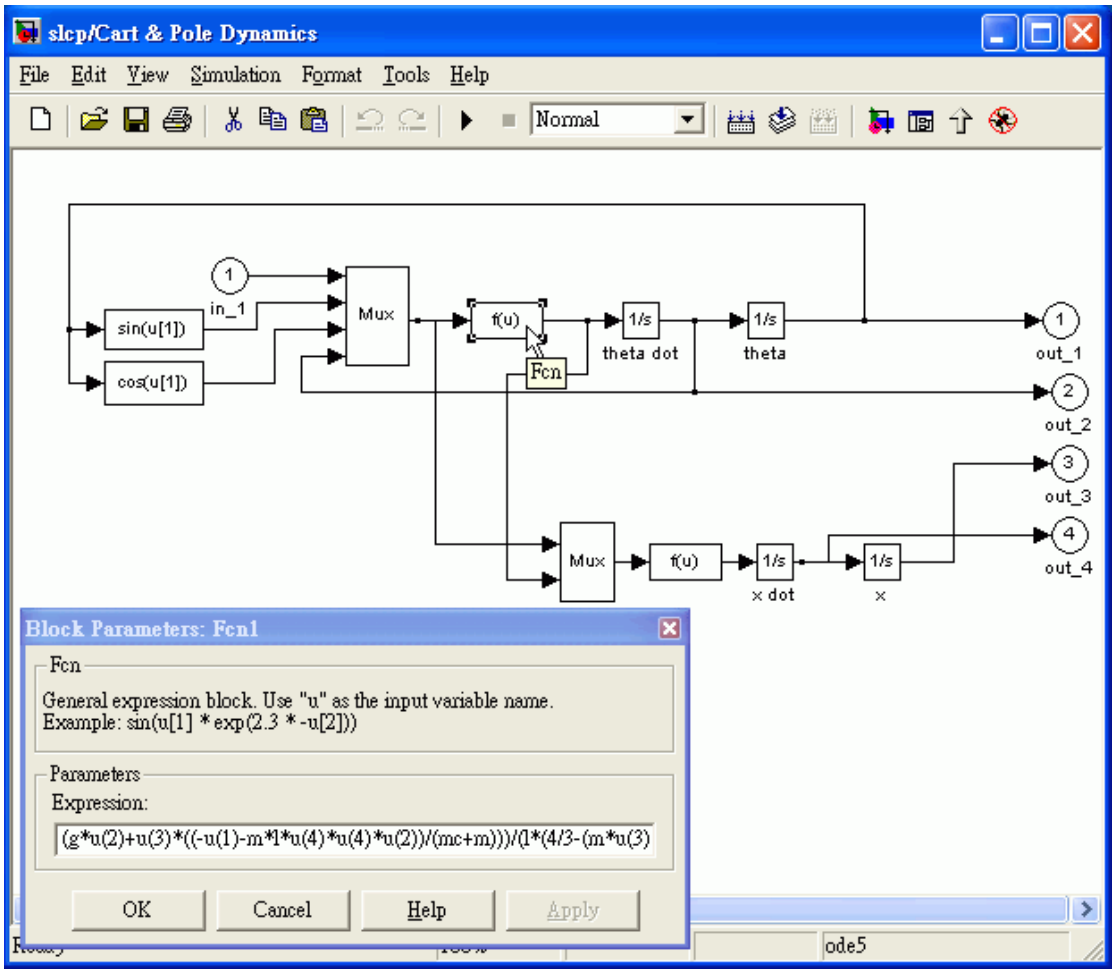
$$\ddot{\theta} = \frac{g\sin(\theta) + \frac{\cos(\theta)(-u - ml\dot{\theta}^2\sin\theta)}{M+m}}{l\left(\frac{4}{3} - \frac{m}{M+m}\cos^2\theta\right)} \qquad (6)$$

$$\ddot{y} = \frac{u + ml\dot{\theta}^2\sin\theta - ml\ddot{\theta}\cos\theta}{M+m} \qquad (7)$$

詳細說明如下

$$\ddot{\theta} = \frac{g\sin(\theta) + \dfrac{\cos(\theta)(-u - ml\dot{\theta}^2\sin\theta)}{M+m}}{l\left(\dfrac{4}{3} - \dfrac{m}{M+m}\cos^2\theta\right)}$$

上方 f(u)之內容



(g*u(2)+u(3)*((-u(1)-m*l*u(4)*u(4)*u(2))/(mc+m)))/(l*(4/3-(m*u(3)*u(3))/(mc+m)))

其中:

$u(1) = u$

$u(2) = \sin\theta$

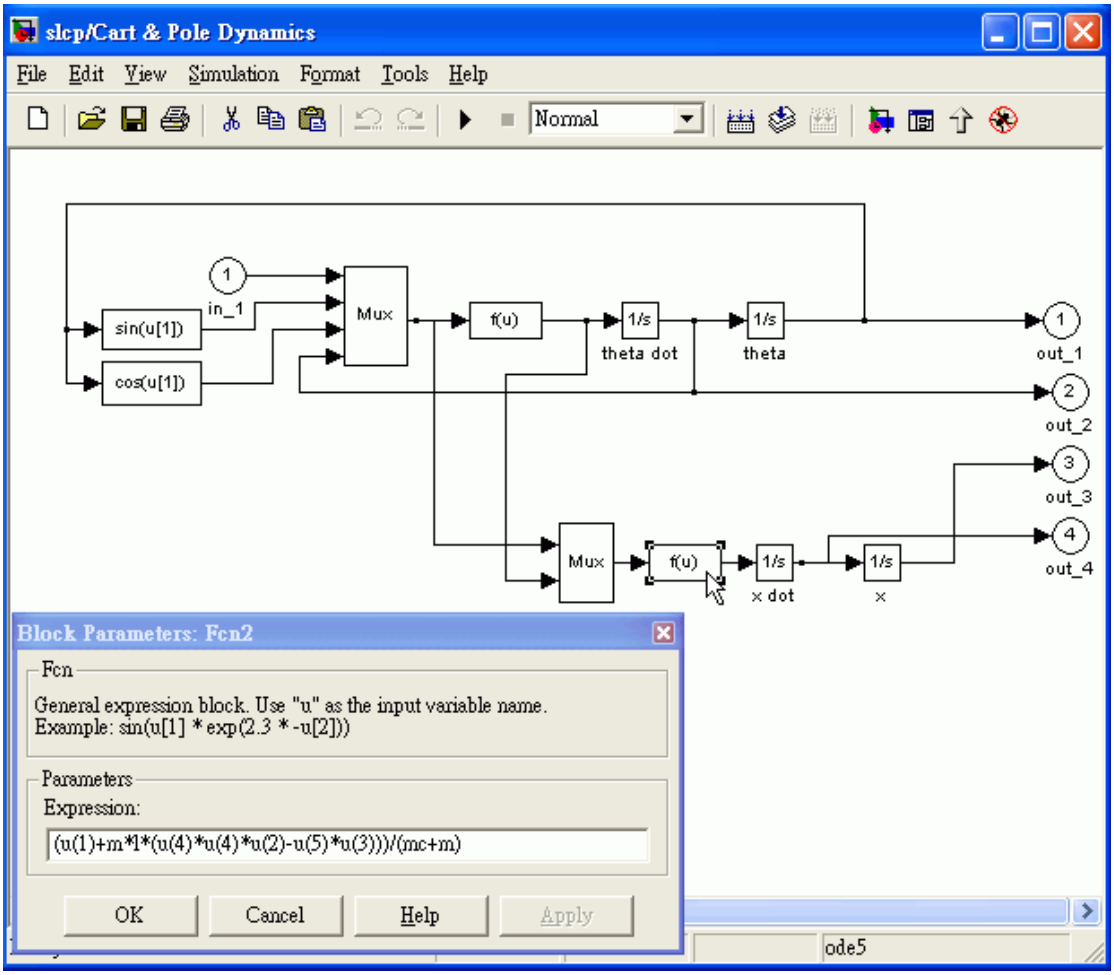
$u(3) = \cos\theta$

$u(4) = \dot{\theta}$

$mc = M$

$$\ddot{y} = \frac{u + ml\dot{\theta}^2 \sin\theta - ml\ddot{\theta}\cos\theta}{M + m}$$

下方 f(u)之內容



(u(1)+m*l*(u(4)*u(4)*u(2)-u(5)*u(3)))/(mc+m)

其中:

$u(1) = u$

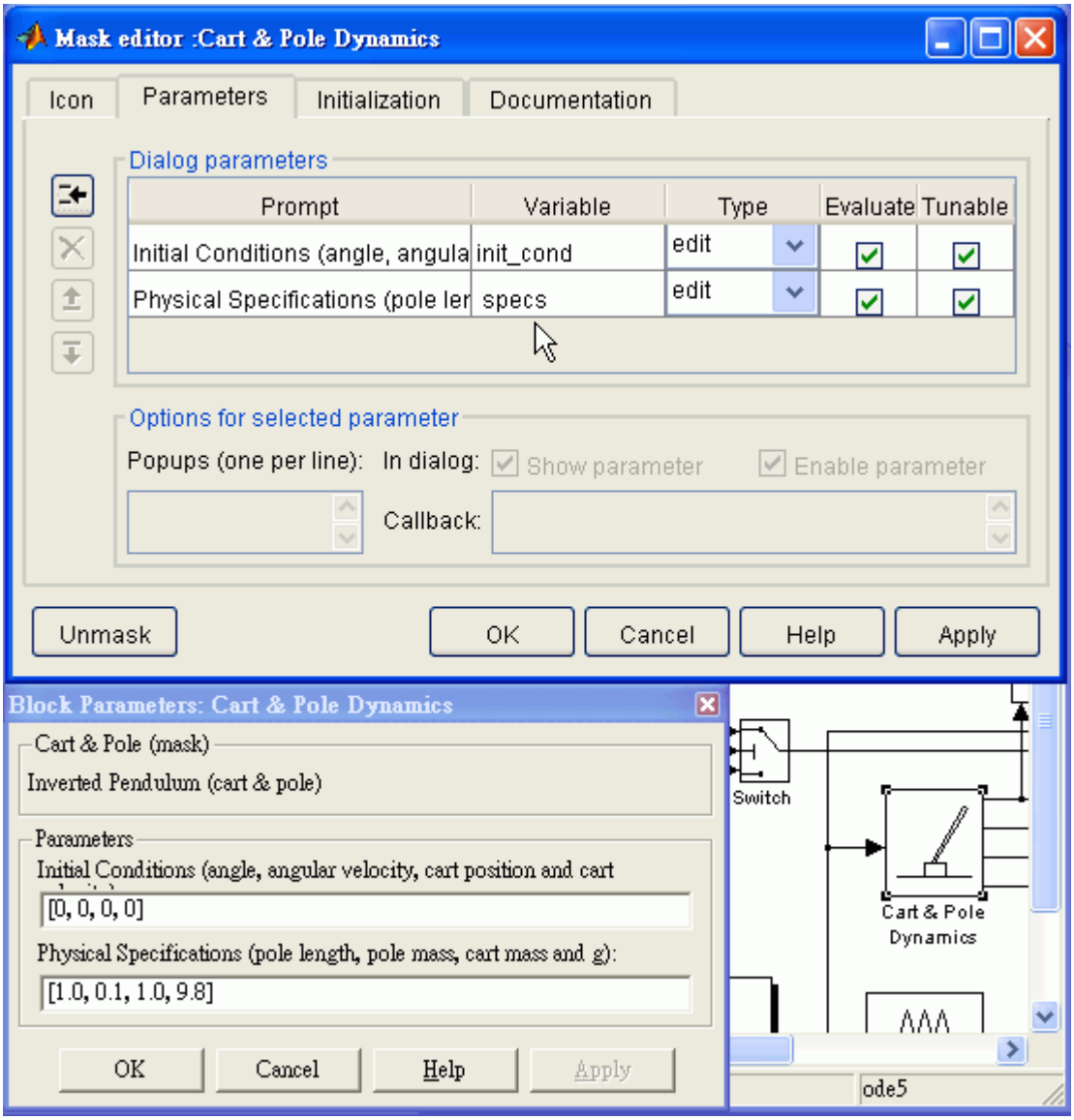
$u(2) = \sin\theta$

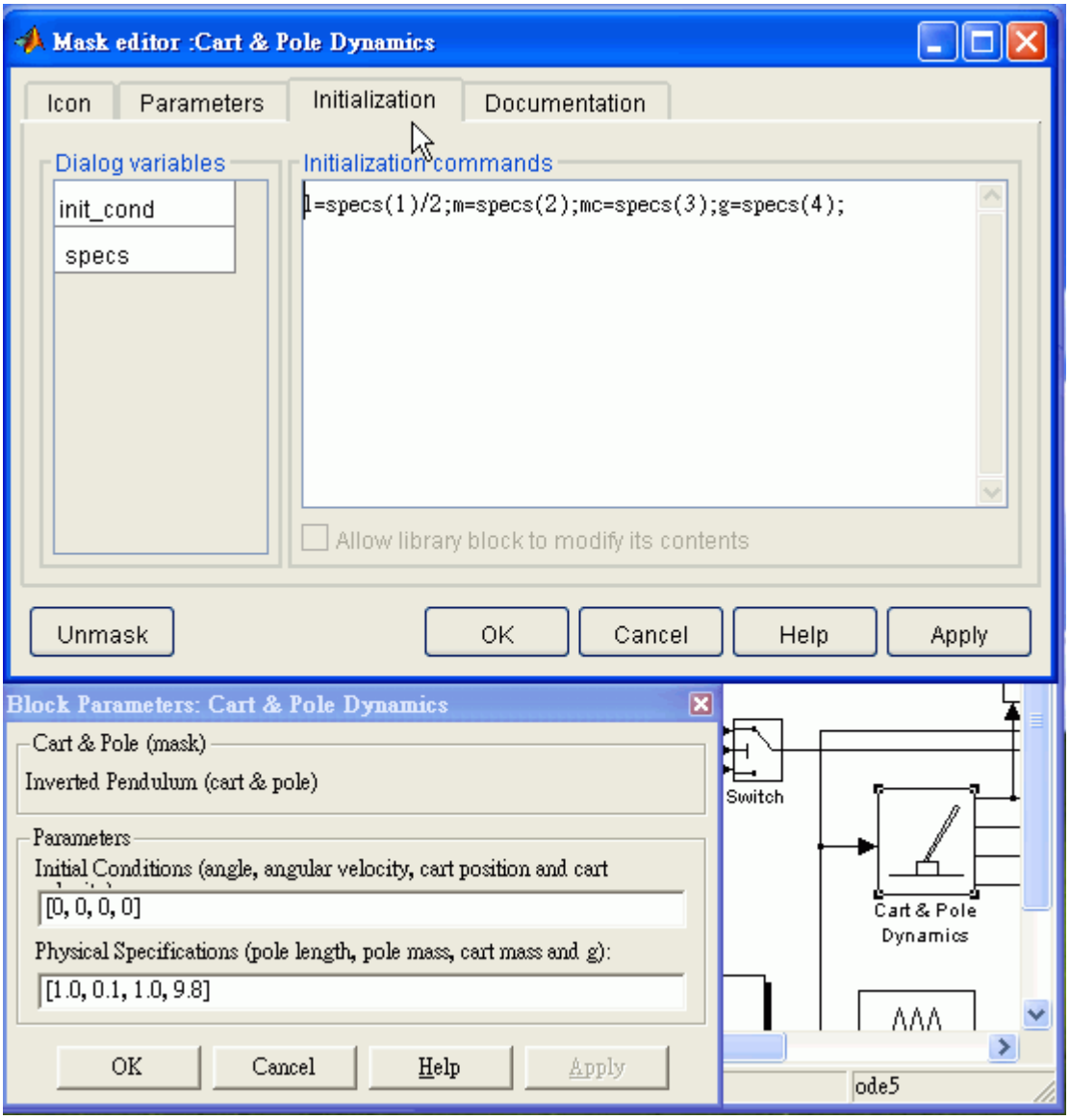
$u(3) = \cos\theta$

$u(4) = \dot{\theta}$

$u(5) = \ddot{\theta}$

$mc = M$


Mask editor :Cart & Pole Dynamics
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Parameters
Initialization
Documentation
Dialog parameters
Prompt
Variable
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Evaluate
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Physical Specifications (pole ler
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Options for selected parameter
Popups (one per line):
In dialog:
Show parameter
Enable parameter
Callback:
Unmask
OK
Cancel
Help
Apply
Block Parameters: Cart & Pole Dynamics
Cart & Pole (mask)
Inverted Pendulum (cart & pole)
Parameters
Initial Conditions (angle, angular velocity, cart position and cart
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Physical Specifications (pole length, pole mass, cart mass and g):
[1.0, 0.1, 1.0, 9.8]
OK
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Switch
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Mask editor :Cart & Pole Dynamics
Icon
Parameters
Initialization
Documentation
Dialog variables
init_cond
specs
Initialization commands
l=specs(1)/2;m=specs(2);mc=specs(3);g=specs(4);
Allow library block to modify its contents
Unmask
OK
Cancel
Help
Apply
Block Parameters: Cart & Pole Dynamics
Cart & Pole (mask)
Inverted Pendulum (cart & pole)
Parameters
Initial Conditions (angle, angular velocity, cart position and cart
[0, 0, 0, 0]
Physical Specifications (pole length, pole mass, cart mass and g):
[1.0, 0.1, 1.0, 9.8]
OK
Cancel
Help
Apply
Switch
Cart & Pole
Dynamics
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